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Convex Polynomial Approximation in L_p $(0<p<1)$*

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We prove that for each convex function $f \in L_p$, $0<p<1$, there exists a convex algebraic polynomial P_n of degree $\leqslant n$ such that

$$\|f-P_n\|_p \leqslant C\omega_2^\varphi\left(f, \frac{1}{n}\right)_p,$$

where $\omega_2^\varphi(f,t)_p$ is the Ditzian–Totik modulus of smoothness of f in L_p, and C depends only on p. Moreover, if f is also nondecreasing, then the polynomial P_n can also be taken to be nondecreasing, thus we have simultaneous monotone and convex approximation in this case.

1. INTRODUCTION

We are interested in the approximation of convex functions $f \in L_p(I)$, $0<p<1$, $I=[-1,1]$, by convex algebraic polynomials. Such approximation has been previously studied by Shvedov [8]. Our main departure from that work is that we prove direct estimates for the error of convex polynomial approximation in terms of the Ditzian–Totik modulus of smoothness. This modulus measures smoothness differently at the end points of I than in the interior, which is crucial (see [5]) if we wish to characterize functions with a given error of polynomial approximation (however, we do not discuss inverse estimates here). In a previous work with X. M. Yu [3], the authors estimated the error of polynomial

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approximation in L_p, $0<p<1$, without constraints and also with a monotonicity constraint. Here, we make use of part of the proof. However, it is interesting to note that the case of monotone approximation required an elaborate construction which in the convex case is much simpler.

If $\varphi(x) := \sqrt{1-x^2}$, then the Ditzian–Totik modulus is defined for each $f \in L_p(I)$ by

$$\omega_k^\varphi(f, t)_p := \sup_{0<h\leqslant t} \left(\int_{-1}^{1} |\Delta_{h\varphi(x)}^k(f, x, I)|^p \, dx \right)^{1/p},$$

where

$$\Delta_{h\varphi(x)}^k(f, x, I) := \begin{cases} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f(x-(k/2)\, h\varphi(x) + ih\varphi(x)), \\ \qquad x \pm (k/2)\, h\varphi(x) \in I, \\ 0, \qquad \text{otherwise}. \end{cases}$$

Then, $\omega_k^\varphi(f, t)_p \leqslant C\|f\|_p(I)$, for sufficiently small $t>0$ (see [5, p. 21]).

We proved in [3] that if $f \in L_p(I)$, $0<p<1$, and $k>0$, then for each $n \geqslant N(p, k)$ there is an algebraic polynomial $P_n(x)$ of degree $\leqslant n$ such that for $C = C(p, k)$,

$$\|f - P_n\|_p \leqslant C\omega_k^\varphi\left(f, \frac{1}{n}\right)_p. \tag{1.1}$$

Moreover for $k \leqslant 2$, the polynomial P_n can be taken to be nondecreasing whenever f is nondecreasing. In this paper, we extend this conclusion to convex approximation.

2. Convex Piecewise Linear Approximants

Let $-1 =: \xi_0 < \xi_1 < \cdots < \xi_n := 1$ be such that adjacent $I_j = [\xi_{j-1}, \xi_j]$, $j = 1, ..., n$, have comparable lengths, i.e., for some $C_0 > 0$

$$\frac{|I_{j\pm 1}|}{|I_j|} \leqslant C_0. \tag{2.1}$$

We are interested in this section in the approximation of $f \in L_p[I]$ by the elements of $\mathscr{S}$ where $\mathscr{S}$ denotes the class of piecewise linear functions on I for this partition.

If $f \in L_p(J)$, $0<p\leqslant 1$, J an interval, then a polynomial P of degree k is a *near best* L_p approximation (with constant M) to f from among all polynomials of degree $\leqslant k$ if

$$\|f - P\|_p(J) \leqslant ME_k(f, J)_p, \tag{2.2}$$

where $E_k(f, J)_p$ is the error of best approximation to f on J in the L_p (quasi-)norm from among all polynomials of degree $\leqslant k$. Of course, if $M=1$ then P is a best approximant.

If P is a near best L_p approximation to f with constant M on an interval J, then (see [2]) it is near best on any larger interval $\tilde{J}$,

$$\|f-P\|_p(\tilde{J}) \leqslant CME_k(f, \tilde{J})_p, \tag{2.3}$$

with C depending only on p, k and the ratio $|\tilde{J}|/|J|$.

Given *any measurable function* f and an interval J, we can speak of a best $L_1(J)$ approximant l to f from linear functions (i.e., polynomial of degree $\leqslant 1$) in the following sense. There should exist a measurable function h with $|h|=1$ on J such that $h(x)=\operatorname{sgn}(f-l)(x)$ whenever $x \in J$ and $|f(x)-l(x)|>0$ and h is orthogonal to all linear functions on J. Brown and Lucier [1] have shown that for each $f \in L_p(J)$, $0<p\leqslant 1$, there exist such linear functions l, and moreover, l is a near best L_p approximant on J.

From now on, $\tilde{I}_j$ denotes the interval with the same center as I_j and twice its length. If $\tilde{I} \subseteq [-1, 1]$, we let l_j be the linear function which interpolates f at ξ_{j-1} and at ξ_j. If f is convex, then l_j is a best L_1 approximation to f in $\tilde{I}_j$ in the above sense. Indeed, if $f \not\equiv l_j$ on I_j, then it is strictly below l_j inside I_j and above it outside I_j. A trivial computation shows that $h := \operatorname{sgn}(f-l_j)$ is orthogonal to all linear functions on $\tilde{I}_j$. Thus, we have

$$\|f-l_j\|_p(\tilde{I}_j) \leqslant ME_1(f, \tilde{I}_j)_p, \tag{2.4}$$

with M depending only on p. We assume that there is at least one interval $\tilde{I}_j \subseteq I$.

Let j_0 be the smallest index such that $\tilde{I}_{j_0} \subseteq [-1, 1]$ and let $I^*_{j_0} := [-1, \xi_{j_0}] \cup \tilde{I}_{j_0}$. Then

$$|\tilde{I}_{j_0}| \leqslant |I^*_{j_0}| \leqslant (|\tilde{I}_{j_0}| + |\tilde{I}_{j_0-1}|) \leqslant (C_0+1)\, |\tilde{I}_{j_0}|. \tag{2.5}$$

It follows by (2.4) and (2.5) that l_{j_0} is near best $L_p(I^*_{j_0})$ approximation to f with a constant CM, where C depends on p and C_0. Similarly we let j_1 be the largest such that $\tilde{I}_{j_1} \subseteq [-1, 1]$ and define $I^*_{j_1}$ accordingly. If for $j_0<j<j_1$ we denote $I^*_j := \tilde{I}_j$, then we have

$$\|f-l_j\|_p \leqslant CME_1(f, I^*_j)_p, \qquad j_0 \leqslant j \leqslant j_1. \tag{2.6}$$

Now define the piecewise linear $S \in \mathscr{S}$ by $S(-1) := l_{j_0}(-1)$, $S(1) := l_{j_1}(1)$, and for $j_0 \leqslant j \leqslant j_1$, $S(\xi_j) := l_j(\xi_j)(=f(\xi_j))$ and linear in between. Then we have proved:

THEOREM 2.1. *For any partition* $-1=\xi_0<\xi_1 \cdots <\xi_n := 1$, *the piecewise linear* $S \in \mathscr{S}$ *given above is continuous and convex on* I *and its linear pieces* l_j *satisfy* (2.6).

For future use, we let $l_j := l_{j_0}$, $0 \leqslant j < j_0$ and $l_j := l_{j_1}$, $j_1 < j \leqslant n$.

In addition to the application of Theorem 2.1 given in the following section, we mention also the following. By choosing $\xi_j = (2j-n)/n$ we have $j_0 = 1$ and $j_1 = n-1$ and S becomes a continuous piecewise linear spline with n equally spaced knots (the usual notation $S \in \mathscr{S}_n^2$). Thus the same proof as in [3] (i.e., an application of Whitney's theorem) gives

COROLLARY 2.2. *If $0 < p < 1$, then for each convex $f \in L_p(I)$, $I = [-1, 1]$, and $n \geqslant 2$ there is a continuous piecewise linear convex spline $S \in \mathscr{S}_n^2$ such that with $C = C(p)$,*

$$\|f - S\|_p \leqslant C\omega_2(f, 1/n)_p \tag{2.7}$$

Moreover, if in addition f is nondecreasing, then S is both nondecreasing and convex.

3. CONVEX POLYNOMIAL APPROXIMATION

We can now state and prove our main result.

THEOREM 3.1. *Let $f \in L_p(I)$, $0 < p < 1$, be convex on I and $k \leqslant 2$. Then, for each $n \geqslant N(p)$, there is a convex algebraic polynomial P_n of degree $\leqslant n$ such that*

$$\|f - P_n\|_p \leqslant C\omega_k^\varphi\left(f, \frac{1}{n}\right)_p. \tag{3.1}$$

Moreover, P_n is nondecreasing and convex on I whenever f is.

Remarks. (i) For $1 \leqslant p \leqslant \infty$ similar estimates for convex functions are due to Leviatan [6] in the case $p = \infty$ (see also [9]), and to Yu [10] and Leviatan and Yu [7] for $1 \leqslant p < \infty$.

(ii) By the aforementioned result of Shvedov [8], (3.1) cannot hold for $k \geqslant 4$ for convex approximation, however, the case $k = 3$ is still open.

Proof. We follow the ideas of [4], as modified in [6], and approximate f by

$$S(x) = l_0 + \sum_{j=0}^{n} c_j \varphi_j, \qquad \varphi_j(x) = (x - \xi_j)_+, \tag{3.2}$$

with $-1 =: \xi_0 < \cdots < \xi_n := 1$ to be selected later, c_j, $j = 1, ..., n-1$ constants, and l_0 a linear function. To approximate S by an algebraic polynomial, we use the Jackson kernel

$$J_n(t) = \lambda_n \left(\frac{\sin nt/2}{\sin t/2}\right)^{2r}, \int_{\pi}^{\pi} J_n(t)\, dt = 1.$$

Here r is a sufficiently large (see the paragraph above (3.3)) fixed natural number. We approximate the characteristic functions $\chi_j(t) := \chi_{[-t_j, t_j]}$, $t_j := j\pi/n$, $j = 0, ..., n$ by the trigonometric polynomials

$$T_j(t) := \chi_j * J_n(t) = \int_{t-t_j}^{t+t_j} J_n(u)\,du, \qquad j = 0, ..., n.$$

The change of variable, $x = \cos t$, gives algebraic polynomials $r_j(x) := T_{n-j}(t)$ which are approximations to the characteristic functions $\chi_{[x_j, 1]}$, $x_j := \cos t_{n-j}$. Finally, we define

$$R_j(x) := \int_{-1}^{x} r_j(u)\,du, \qquad j = 0, ..., n,$$

which can be viewed as approximations to the truncated power functions φ_j. Note that $R_0(x) = 1 + x$ and $R_n(x) \equiv 0$ and in general R_j is a polynomial of degree $\leqslant nr$. Also $r_j - r_{j+1} \geqslant 0$, $x \in I$, and therefore $R_j - R_{j+1}$ is increasing on I for all $j = 0, 1, ..., n-1$. We now choose the partition points ξ_j so that $1 - \xi_j = R_j(1)$. Then $-1 = \xi_0 < \xi_1 < \cdots < \xi_n = 1$. It was proved in [6] that R_j is convex in I.

The piecewise linear function S of Theorem 2.1 is convex and can be represented as in (3.2) with $c_j = a_j - a_{j-1}$ with a_j the slope of l_j (and hence $a_j \geqslant a_{j-1}$). Replacing φ_j by R_j, we get a convex polynomial

$$P_n(f) := l_0 + \sum_{j=0}^{n-1} (a_j - a_{j-1})\,R_j.$$

Indeed, $P_n(f)$ is convex as it is a sum of convex polynomials with nonnegative coefficients. Now the same proof as in [3] (we omit the details) gives for suitably large r

$$\|f - S\|_p \leqslant C\omega_2^{\varphi}\left(f, \frac{1}{n}\right)_p. \tag{3.3}$$

and

$$\|S - P_n\|_p \leqslant C\omega_2^{\varphi}\left(f, \frac{1}{n}\right)_p. \tag{3.4}$$

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